Recitation 11

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Review

Orthogonal matrix: The following are equivalent:

- An $n \times n$ matrix U is orthogonal;
- $U^T U = I$;
- $U^{-1} = U$;
- columns of U form an orthoNORMAL basis.

Projection to a subspace: if $W \subset \mathbb{R}^n$ is a subspace, and $y \in \mathbb{R}^n$ is a vector, to find $proj_W(y)$ you need to

- Find orthogonal basis $\{u_1, \dots, u_m\}$ of W.
- Then $proj_W(y) = c_1 u_1 + \dots + c_m u_m$, where $c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$.

If you already know an orthoNORMAL basis $v_1, \dots, v_m$ of W: put $U = [v_1 \dots v_m]$, then $proj_W(y) = UU^T y$.

Minimal distance: if $y \in \mathbb{R}^n$, minimal distance from y to $W \subset \mathbb{R}^n$ is the length $||y - proj_W(y)||$. The **closest point**, or **best approximation** to y in W turns out to be the projection $proj_W(y)$ (what a surprise!).
Finding orthogonal basis: suppose you want to find an orthogonal basis in $W \subset \mathbb{R}^n$. To do that,

- Find some (any) basis $x_1, \dots, x_m$ of W.
- Orthogonalize it using Gramm-Schmidt to get an orthogonal basis $v_1, \dots, v_m$ of W.

Finding orthoNORMAL basis:

- Find orthogonal basis.
- Normalize, i.e. rescale each vector to make it have length 1.

Gramm-Schmidt: Let $x_1, \dots, x_m$ be a basis in $W \subset \mathbb{R}^n$. You want to find a new orthogonal basis using the old maybe-not-orthogonal one. You can use formulas:

$$\begin{aligned}
v_1 &= x_1 \\
v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\
v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\
\dots &= \dots \\
v_m &= x_m - \frac{x_m \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_m \cdot v_2}{v_2 \cdot v_2} v_2 - \dots - \frac{x_m \cdot v_{m-1}}{v_{m-1} \cdot v_{m-1}} v_{m-1}
\end{aligned}$$

QR-factorization: If A is an $n \times m$ matrix with m independent columns, then $A = QR$, where Q is $n \times m$ with columns forming an orthonormal basis of $Col(A)$, and R is $m \times m$ upper triangular matrix with positive entries on the diagonal. **How to find all that?**
If $x_1, \dots, x_m$ are columns of A, apply Gramm-Schmidt, get $v_1, \dots, v_m$. These would be columns of Q. Then $R = Q^T A$.

Problems

Problem 1. Find the distance from $y = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ to the line W in $\mathbb{R}^3$ given by the equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

What is the point on W closest to y? (Hint: find an orthogonal basis of L and do the projection business).

Problem 2. Find the projection of the vector $y_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ onto the plane $W \subset \mathbb{R}^3$ given by the equation $x_1 - 3x_2 - x_3 = 0$.

Find the projections of $y_2 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ and of $y_3 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ onto W. Analyze your results. (Hint: you can either guess an orthogonal basis of W, or you can find any basis and then Gramm-Schidt it.)

Problem 3. OrthoNORMALIZE the set of vectors $x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$

Problem 4. Find a QR-decomposition of $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 1 & 4 & 2 \\ -1 & 0 & 1 \end{bmatrix}$. (Hint: use the previous exercise.)

Problem 5. Let W be the space $W = Col(A) \subset \mathbb{R}^4$, where A is the matrix above. Find the projection of $y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto W. (Hint: use what you have found before. It would be nice (?) to use the map $v \mapsto UU^T v$.)

Problem 6. Show that Gramm-Schmidt doesn't do anything if you start from already an orthogonal basis.

Problem 7. Let $W \subset \mathbb{R}^3$ be a plane $x_1 - x_2 + x_3 = 0$, and let $T \colon \mathbb{R}^3 \to \mathbb{R}^{\nVDash}$ be the transformation $v \mapsto T(v) := proj_W(v)$.

- Without doing any calculation, explain why $T \circ T = T$, i.e. applying T twice is the same as applying it once.
- Find a basis of W. Call it $\{v_1, v_2\}$.
- Let $v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Explain why $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
- Find the matrix of T relative to this basis.
- Analyze what you've got.

Problem 8. Let A be an $m \times n$ matrix. Prove that any vector x in $\mathbb{R}^n$ can be written uniquely as a sum $x = p + u$, where $p \in Row(A)$ and $u \in Nul(A)$. (Hint: what is $Row(A)^{\perp}$?)

Prove that if $Ax = b$ is consistent for some $b \in \mathbb{R}^m$, then there is a unique $p \in Row(A)$ s.t. $Ap = b$.